Subcritical branching processes in random environment without Cramer condition

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Abstract

A subcritical branching process in random environment (BPRE) is considered whose associated random walk does not satisfy the Cramer condition. The asymptotics for the survival probability of the process is investigated, and a Yaglom type conditional limit theorem is proved for the number of particles up to moment n given survival to this moment. Contrary to other types of subcritical BPRE, the limiting distribution is not discrete. We also show that the process survives for a long time owing to a single big jump of the associate random walk accompanied by a population explosion at the beginning of the process.

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Branching process, random environment, random walk, survival probability, functional limit theorem
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1. Introduction and Main Results

In this paper we consider the asymptotic behavior of a type of subcritical branching processes in random environment. More specifically, the random environment is given by a sequence of independent and identically distributed (i.i.d.) probability distributions on nonnegative integers, denoted by $\pi = \{\pi_n,\ n \geq 0\}$ where

$$\pi_n = \left\{\pi_n^{(0)}, \pi_n^{(1)}, \pi_n^{(2)}, ...\right\},\ \pi_n^{(i)} \geq 0,\ \sum_{i=0}^{\infty} \pi_n^{(i)} = 1,$$

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which are defined on a common probability space $(\Omega, \mathcal{A}, \mathbf{P})$. Moreover, for a given environment $\pi = \{\pi_n\}$, the branching process $\{Z_n,\ n \geq 0\}$ satisfies

$$Z_0 = 1,\ \mathbf{E}_\pi \left[s^{Z_{n+1}} |\ Z_0, Z_1, ..., Z_n \right] = \left(f_n\left(s\right)\right)^{Z_n},$$

where $f_n\left(s\right) = \sum_{i=0}^{\infty} \pi_n^{(i)} s^i$ is the generating function of π_n; in other words, π_n is the (common) offspring distribution for the particles at generation n. Here and below we use the subscript π to indicate that the expectation (or probability with the notation $\mathbf{P}_\pi$) is taken under the given environment π. As is shown in various articles on the branching processes in random environment, the asymptotic behavior of $\{Z_n\}$ is crucially affected by the so-called *associated random walk* $\{S_n\}$ defined as follows:

$$S_0 = 0,\ S_n := X_1 + X_2 + \cdots + X_n,\ n \geq 1,$$

where

$$X_n = \log f_{n-1}^{\prime}\left(1\right), n = 1, 2, \ldots.$$

are the logarithmic mean offspring numbers. For notational ease let

$$f\left(s\right) = f_0\left(s\right), \quad \text{and} \quad X = X_1.$$

We call $\{Z_n\}$ a *subcritical* branching process in random environment if

$$\mathbf{E}\left[X\right] =: -a < 0. \tag{1}$$

By the SLLN this implies that the associated random walk $\{S_n\}$ diverges to $-\infty$ almost surely, which, in view of the inequality $\mathbf{P}_\pi\left(Z_n > 0\right) \leq \mathbf{E}_\pi\left[Z_n\right] = e^{S_n}$, leads to almost sure extinction of $\{Z_n\}$.

Subcritical branching processes in i.i.d. random environment have been considered in a number of articles, see, for instance, Afanasyev (1980), Dekking (1988), Liu (1996), Souza and Hambly (1997), Afanasyev (1998), Fleischmann and Vatutin (1999), Afanasyev (2001), Guivarc'h and Liu (2001), Geiger et al. (2003), Vatutin (2003), Afanasyev et al. (2005), Bansaye (2008), Bansaye (2009), Afanasyev et al. (2010), and Afanasyev et al. (2011). According to these papers, a subcritical branching process in random environment is called *weakly* subcritical if there exists $\theta \in (0, 1)$ such that $\mathbf{E}\left[Xe^{\theta X}\right] = 0$; *intermediately* subcritical if $\mathbf{E}\left[Xe^{X}\right] = 0$; and *strongly* subcritical if $\mathbf{E}\left[Xe^{X}\right] < 0$.

The classification above is not exhaustive, though. An important exception is that the random variable X is such that $\mathbf{E}\left[Xe^{\theta X}\right] = \infty$ for any $\theta > 0$, and this is the focus of the present paper. To be more specific, we suppose that $\sigma^2 := \mathrm{Var}(X) < \infty$ and, in addition, following the custom of writing $f \sim g$ to mean that the ratio f/g converges to 1, we have, as $x \to \infty$,

$$A(x) := \mathbf{P}\left(X > x\right) \sim \frac{l(x)}{x^\beta}, \quad \text{for some } \beta > 2, \tag{2}$$

where $l(x)$ is a function slowly varying at infinity. Thus, the random variable X does not satisfy the Cramer condition.

We will see below that for the non-Cramer case, similarly to other cases, the asymptotics of the survival probability and the growth of the population size given survival are specified in main by the behavior of the associated random walk. However, the influence of the associated random walk for the non-Cramer case has essentially different nature: for the Cramer cases, the survival for a long time happens due to the "atypical" behavior of the *whole* trajectory of the associated random walk that results in its smaller, then usually, slope for the strongly subcritical case (Afanasyev et al. (2005)), in its convergence to a Levy process attaining its minimal value at the end of the observation interval for the intermediately subcritical case (Afanasyev et al. (2011)), and in the positivity of its essential part for the weakly subcritical case (Afanasyev et al. (2010)). For the non-Cramer case, the process survives for a long time owing to a *single* big jump of the associated random walk at the beginning of the evolution which, in turn, is accompanied by an explosion of the population size at this moment; see Lemmas 14 and 15 for the precise information. Besides, the number of particles at a distant moment n given its survival up to this moment tends to infinity for the non-Cramer case, while for the other types of subcritical processes in random environment such conditioning leads to discrete limiting distributions with no atoms at infinity.

One of our assumptions is the following (technical) condition for $A(x)$: for any fixed $h > 0$,

$$A(x+h) - A(x) = -\frac{h\beta A(x)}{x}(1+o(1)) \text{ as } x \to \infty. \tag{3}$$

Next, for any offspring distribution $\widetilde{\pi} = \{\widetilde{\pi}^{(i)} : i \geq 0\}$ with generating function $\widetilde{f}(s)$, denote

$$\eta(\widetilde{\pi}) = \frac{\sum_{i=0}^{\infty} i(i-1)\widetilde{\pi}^{(i)}}{2(\sum_{i=0}^{\infty} i\widetilde{\pi}^{(i)})^2} = \frac{\widetilde{f}''(1)}{2\left(\widetilde{f}'(1)\right)^2}. \tag{4}$$

Introduce the following

Assumption 1. *(i) There exists $\delta > 0$ such that, as $x \to \infty$,*

$$\mathbf{P}\left(\eta(\pi_0) > x\right) = o\left(\frac{1}{\log x \times (\log\log x)^{1+\delta}}\right).$$

(ii) As $x \to \infty$, (under probability $\mathbf{P}$,)

$$\mathcal{L}\left(f\left(1 - e^{-x}\right) | X > x\right) \Longrightarrow \mathcal{L}(\gamma), \tag{5}$$

where γ is a random variable which is less than 1 with a positive probability.

It can be shown that if π_0 is either almost surely a Poisson distribution or almost surely a geometric distribution, and if (2) is satisfied, then (5) holds with $\gamma \equiv 0$. Moreover, it is not difficult to give an example of branching processes in random environment where γ is either positive and less than 1 with probability 1, or random with support not concentrated at 1. Indeed, let γ be a random variable taking values in $[0, 1-\delta] \subset [0,1]$ for some $\delta \in (0,1]$, and p and $q, p+q=1, pq>0$, be random variables independent of γ such that the random variable $X := \log(1-\gamma) + \log(p/q)$ meets conditions (1) and (2). Define $f(s) := \gamma + (1-\gamma)\, q/(1-ps)$. Then $f'(1) = (1-\gamma)\, p/q = \exp(X)$, and it is straightforward to show that for any $\varepsilon \in (0,1)$,

$$\lim_{x\to\infty} \mathbf{P}\left(|f(1-e^{-x}) - \gamma| \geq \varepsilon \,|\, X > x\right) \leq \lim_{x\to\infty} \mathbf{P}\left(X - x \leq -\log\varepsilon \,|\, X > x\right) = 0,$$

therefore (5) holds.

Let us briefly explain Assumption 1(ii). For any fixed environment π and any $x > 0$, let $\mathcal{L}_\pi\left(Z_1 e^{-x}\right)$ be the distribution of $Z_1 e^{-x}$. Note that this actually only depends on π_0. We will show in Lemma 9 that if condition (2) holds, then (5) is equivalent to the following assumption, concerning weak convergence of *random* measures: (under probability $\mathbf{P}$,)

$$\text{conditional on } \{X > x\},\ \mathcal{L}_\pi\left(Z_1 e^{-x}\right) \Longrightarrow \gamma\delta_0 + (1-\gamma)\,\delta_\infty \quad \text{as } x \to \infty,$$

where δ_0 and δ_∞ are measures assigning unit masses to the corresponding points.

In what follows we assume that *the distribution of X is nonlattice.* The case when the distribution of X is lattice needs natural changes related to the local limit theorem that we use in our proofs (see Proposition 5 below).

Define

$$f_{k,n}(s) := f_k(f_{k+1}(...(f_{n-1}(s))...)),\ 0 \leq k \leq n-1, \quad \text{and} \quad f_{n,n}(s) := s. \tag{6}$$

When $k = 0$, $f_{0,n}(s) = E_\pi(s^{Z_n})$ is the conditional probability generating function of Z_n.

The following is our first main theorem which deals with the the survival probability of the process.

Theorem 2. *Assume conditions* (1), (2), (3) *and Assumption* 1. *Then the survival probability of the process $\{Z_n\}$ has, as $n \to \infty$, the asymptotic representation*

$$\mathbf{P}(Z_n > 0) \sim K\mathbf{P}(X > na),$$

where

$$K := \sum_{j=0}^{\infty} \mathbf{E}\left[1 - f_{0,j}(\gamma)\right] \in (0, \infty), \tag{7}$$

and γ is a random variable that has the same distribution as the γ in Assumption 1(*ii*) *and is independent of the underlying environment $\{\pi_n\}$ (and consequently of $\{f_{0,j}\}$).*

As we mentioned earlier, if π_0 is either almost surely a Poisson distribution or almost surely a geometric distribution, then (2) implies (5) with $\gamma \equiv 0$, so the constant K becomes $\sum_{j=0}^{\infty} \mathbf{P}(Z_j > 0)$. In this case we can give the following intuitive explanation of Theorem 2: Let

$$U_n = \inf \{j : X_j > na\} \tag{8}$$

be the first time when the increment of the random walk $S := \{S_j, j \geq 0\}$ exceeds na. Then one can show that the event $\{Z_n > 0\}$ is asymptotically equivalent to $\{U_n < n, Z_{U_n - 1} > 0\} = \cup_{j<n}\{Z_{j-1} > 0, U_n = j\}$. Now for each fixed $j \geq 1$, $\mathbf{P}(Z_{j-1} > 0, U_n = j) \sim \mathbf{P}(Z_{j-1} > 0)\mathbf{P}(X > na)$, and hence, not rigorously,

$$\mathbf{P}(Z_n > 0) \sim \mathbf{P}(U_n < n, Z_{U_n-1} > 0) \sim \sum_{j=1}^{\infty} \mathbf{P}(Z_{j-1} > 0) \cdot \mathbf{P}(X > na) = K\mathbf{P}(X > na).$$

In fact, we may say more: (even in the general case when $\gamma \not\equiv 0$,) the process survives owing to one big jump of the associated random walk which happens at the very beginning of the evolution of the process; moreover, the big jump is accompanied by a population explosion which leads to survival. See Lemmas 14 and 15 for the precise information.

The next result gives a Yaglom type conditional limit theorem for the number of particles up to moment n given survival of the process to this moment. Recall that $\sigma^2 = \mathrm{Var}(X)$, and U_n is defined in (8).

Theorem 3. *Assume conditions* (1), (2), (3) *and Assumption* 1. *Then for any* $j \geq 1$,

$$\lim_{n\to\infty} \mathbf{P}(U_n = j \mid Z_n > 0) = \mathbf{E}(1 - f_{0,j-1}(\gamma))/K.$$

Moreover,

$$\mathcal{L}\left(\frac{Z_{[nt]\vee U_n}}{Z_{U_n} \exp(S_{[nt]\vee U_n} - S_{U_n})},\ 0 \leq t \leq 1 \,\middle|\, Z_n > 0 \right) \Longrightarrow (1,\ 0 \leq t \leq 1)\,, \tag{9}$$

$$\mathcal{L}\left(\frac{1}{\sigma\sqrt{n}} \left(\log\left(Z_{[nt]\vee U_n}/Z_{U_n}\right) + nta\right),\ 0 \leq t \leq 1 \,\middle|\, Z_n > 0 \right) \Longrightarrow (B_t,\ 0 \leq t \leq 1),$$

and for any $\varepsilon > 0$,

$$\mathcal{L}\left(\frac{1}{\sigma\sqrt{n}} \left(\log\left(Z_{[nt]}/Z_{[n\varepsilon]}\right) + n(t-\varepsilon)a\right),\ \varepsilon \leq t \leq 1 \,\middle|\, Z_n > 0 \right) \Longrightarrow (B_t - B_\varepsilon,\ \varepsilon \leq t \leq 1),$$

where the symbol $\Longrightarrow$ *means weak convergence in the space* $D[0,1]$ *or* $D[\varepsilon,1]$ *endowed with Skorokhod topology, and* B_t *is a standard Brownian motion.*

Therefore after the population explosion at time U_n, the population drops exponentially at rate a, with a fluctuation of order $\exp(O(\sqrt{k}))$ with k the number of generations

elapsed after the explosion. Moreover, it follows from (9) and the continuous mapping theorem that

$$\mathcal{L}\left(\log\left(Z_{[nt]\vee U_n}/Z_{U_n}\right)-(S_{[nt]\vee U_n}-S_{U_n}),0\le t\le 1\middle|\,Z_n>0\right)\Longrightarrow(0,0\le t\le 1),$$

and, therefore, after the big jump, at the logarithmic level the fluctuations of the population are *completely* described by the fluctuations of the associated random walk.

2. Some Preliminary Results

We list some known results for the random walk S and establish some new ones.

Define

$$M_n=\max_{1\le k\le n}S_k,\quad L_n=\min_{0\le k\le n}S_k,\ \tau_n=\min\left\{0\le k\le n:S_k=L_n\right\},$$

$$\tau(x)=\inf\left\{k>0:\ S_k<-x\right\},\quad x\ge 0,$$

and $\tau=\tau(0)=\inf\left\{k>0:\ S_k<0\right\}$. Further, let

$$D:=\sum_{k=1}^{\infty}\frac{1}{k}\mathbf{P}\left(S_k\ge 0\right),$$

which is clearly finite given conditions (1) and (2).

Proposition 4. [*Borovkov and Borovkov (2008, Theorems 8.2.4, page 376)*] *Under conditions* (1) *and* (2), *as* $n\to\infty$,

$$\mathbf{P}\left(L_n\ge 0\right)=\mathbf{P}\left(\tau>n\right)\sim e^{D}\mathbf{P}\left(X>an\right).$$

Next, let $Y=X+a$. Then Y is a random variable with nonlattice distribution, and with zero mean and finite variance. Moreover, as $x\to\infty$, the function

$$B(x):=\mathbf{P}(Y>x)\ (=\mathbf{P}(X>x-a)=A(x-a))\ \sim\frac{l(x)}{x^{\beta}},\quad\beta>2,$$

and satisfies a modified version of (3) by replacing $A(x)$ with $B(x)$.

Proposition 5. [*Borovkov and Borovkov (2008, Theorem 4.7.1, page 218)*] *Assume* (2) *and* (3). *Then with* $\tilde{S}_n:=Y_1+...+Y_n$, *where* $Y_i\overset{d}{=}Y$ *and independent, we have for any* $h>0$, *uniformly in* $x\ge N\sqrt{n\log(n+1)}$, *as* $N\to\infty$,

$$\mathbf{P}\left(\tilde{S}_n\in[x,x+h)\right)=\frac{h\beta nB(x)}{x}(1+o(1)).$$

The uniformity of $o(1)$ above is understood as that there exists a function $\delta(N) \downarrow 0$ as $N \to \infty$ such that the term $o(1)$ could be replaced by a function $\delta_h(x,n)$ with $|\delta_h(x,n)| \le \delta(N)$.

Based on Propositions 4 and 5 we prove the following

Lemma 6. *Assume conditions* (1), (2) *and* (3). *Then, as* $n \to \infty$,

$$\mathbf{E}\left[e^{S_n}; \tau_n = n\right] = \mathbf{E}\left[e^{S_n}; M_n < 0\right] \sim \frac{K_1}{n}\mathbf{P}\left(X > an\right),$$

where

$$K_1 := \frac{\beta}{a}\exp\left\{\sum_{n=1}^{\infty}\frac{1}{n}\mathbf{E}\left[e^{S_n}; S_n < 0\right]\right\} < \infty. \tag{10}$$

Proof. The first equality follows from duality. More specifically, the random walks $\{S_k : k = 0, 1, \ldots, n\}$ and $\{S'_k := S_n - S_{n-k} : k = 0, 1, \ldots, n\}$ have the same law, and the event $\{\tau_n = n\}$ for $\{S_k\}$ corresponds to the event $\{M'_n < 0\}$ for $\{S'_k\}$.

Next we evaluate the quantity

$$\mathbf{E}\left[e^{S_n}; S_n < 0\right] = \mathbf{E}\left[e^{S_n}; -(\beta+2)\log n \le S_n < 0\right] + O\left(n^{-\beta-2}\right). \tag{11}$$

Clearly, for any $h > 0$,

$$\begin{aligned}
&\sum_{0 \le k \le (\beta+2)h^{-1}\log n} e^{-(k+1)h} \cdot \mathbf{P}\left(-(k+1)h + an \le \tilde{S}_n \le -kh + an\right) \\
\le\ & \mathbf{E}\left[e^{S_n}; -(\beta+2)\log n \le S_n < 0\right] \\
\le\ & \sum_{0 \le k \le (\beta+2)h^{-1}\log n} e^{-kh} \cdot \mathbf{P}\left(-(k+1)h + an \le \tilde{S}_n \le -kh + an\right).
\end{aligned}$$

By Proposition 5, in the range of k under consideration, as $n \to \infty$,

$$\begin{aligned}
\mathbf{P}\left(-(k+1)h + an \le \tilde{S}_n \le -kh + an\right) &= \frac{h\beta n}{(-(k+1)h + an)}B\left(-(k+1)h + an\right)(1 + o(1)) \\
&= \frac{h\beta}{a}A\left(an\right)(1 + o(1)),
\end{aligned}$$

where $o(1)$ is uniform in $0 \le k \le (\beta+2)\,h^{-1}\log n$. Now passing to the limit as $n \to \infty$ we get

$$\begin{aligned}
h\sum_{k=0}^{\infty} e^{-(k+1)h} &\le \liminf_{n\to\infty} \frac{a\mathbf{E}\left[e^{S_n}; -(\beta+2)\log n \le S_n < 0\right]}{\beta A\left(an\right)} \\
&\le \limsup_{n\to\infty} \frac{a\mathbf{E}\left[e^{S_n}; -(\beta+2)\log n \le S_n < 0\right]}{\beta A\left(an\right)} \\
&\le h\sum_{k=0}^{\infty} e^{-kh}.
\end{aligned}$$

Letting now $h \to 0+$ we see that

$$\lim_{n\to\infty} \frac{a\mathbf{E}\left[e^{S_n}; -(\beta+2)\log n \le S_n < 0\right]}{\beta A(an)} = 1.$$

Combining this with (11) we conclude that, as $n \to \infty$,

$$\mathbf{E}\left[e^{S_n}; S_n < 0\right] = \frac{\beta}{a} A(an)(1+o(1)) \sim \frac{\beta}{a}\mathbf{P}(X > an). \tag{12}$$

Furthermore, we know by a Baxter identity that

$$\exp\left\{\sum_{n=1}^{\infty} \frac{t^n}{n}\mathbf{E}\left[e^{S_n}; S_n < 0\right]\right\} = 1 + \sum_{n=1}^{\infty} t^n \mathbf{E}\left[e^{S_n}; M_n < 0\right],$$

see for example Chapter XVIII.3 in Feller (1966) or Chapter 8.9 in Bingham et al. (1987). From (12) and Theorem 1 in Chover et al. (1973) we get

$$\mathbf{E}\left[e^{S_n}; M_n < 0\right] \sim \frac{K_1}{n}\mathbf{P}(X > an),$$

where K_1 is given by (10). That $K_1 < \infty$ follows from (12). $\square$

Corollary 7. *Under the conditions of Lemma* 6*, the constant* K *in* (7) *is finite.*

Proof. Clearly,

$$1 - f_{0,j}(\gamma) \le 1 - f_{0,j}(0) = \mathbf{P}_\pi(Z_j > 0) = \min_{0\le i\le j} \mathbf{P}_\pi(Z_i > 0) \le \min_{0\le i\le j} e^{S_i} = e^{S_{\tau_j}}. \tag{13}$$

Thus

$$\begin{aligned} K &= \sum_{j=0}^{\infty} \mathbf{E}\left[1 - f_{0,j}(\gamma)\right] \le \sum_{j=0}^{\infty} \mathbf{E}\left[e^{S_{\tau_j}}\right] \\ &= \sum_{j=0}^{\infty}\sum_{i=0}^{j} \mathbf{E}\left[e^{S_i}; \tau_j = i\right] = \sum_{i=0}^{\infty} \mathbf{E}\left[e^{S_i}; \tau_i = i\right] \cdot \sum_{j=i}^{\infty} \mathbf{P}(L_{j-i} \ge 0). \end{aligned}$$

The last term is finite by Lemma 6 and Proposition 4. $\square$

Next, recall that $U_n = \inf\{j : X_j > na\}$, and $\tau = \inf\{j > 0 : S_j < 0\}$. The next result says that if the associated random walk remains nonnegative for a long time, then there must be a big jump at the beginning.

Proposition 8. [*Durrett (1980, Theorem 3.2, page 283)*] *If conditions* (1) *and* (2) *hold then*

$$\lim_{n\to\infty} \mathbf{P}(U_n = j | \tau > n) = \frac{1}{\mathbf{E}\tau}\mathbf{P}(\tau > j - 1).$$

3. Proof of Theorem 2

We first introduce the following convergence statements, which will be shown to be equivalent to each other: (under probability $\mathbf{P}$,) as $x \to \infty$,

(i) for any function $\delta(x)$ satisfying $\lim_{x\to\infty} \delta(x) = 0$, $\mathcal{L}\left(f\left(1 - \exp(-x(1+\delta(x)))\right) | X > x\right) \Longrightarrow \mathcal{L}(\gamma)$;

(ii) for any function $\delta(x)$ satisfying $\lim_{x\to\infty} \delta(x) = 0$, $\mathcal{L}\left(f\left(\exp(-\exp(-x(1+\delta(x))))\right) | X > x\right) \Longrightarrow \mathcal{L}(\gamma)$;

(iii) for any $\lambda > 0$, $\mathcal{L}\left(f\left(\exp\left(-\lambda \exp(-x)\right)\right) | X > x\right) \Longrightarrow \mathcal{L}(\gamma)$; and

(iv) conditional on $\{X > x\}$, $\mathcal{L}_\pi\left(Z_1 e^{-x}\right) \Longrightarrow \gamma\delta_0 + (1-\gamma)\,\delta_\infty$.

Lemma 9. *Assume condition* (2). *Then the convergences* (i) ∼(iv) *above are equivalent, and are all equivalent to* (5).

Proof. We will show that (5)⇒(i)⇒(ii)⇒(iii)⇔ (iv), and finally (iii)⇒(5).

We first prove that (5) implies (i). By (2), the events $\{X > x\}$ and $\{X > x(1+\delta(x))\}$ are asymptotically equivalent to each other (in the sense that $\lim_{x\to\infty} \mathbf{P}(X > x | X > x(1+\delta(x)) = \lim_{x\to\infty} \mathbf{P}(X > x(1+\delta(x)) | X > x) = 1$), hence it follows, for example, from Lemma 17 in Lalley and Zheng (2011), that

$$\begin{aligned}\lim_{x\to\infty} \mathbf{P}\left(f(1 - e^{-x(1+\delta(x))}) > y | X > x\right) &= \lim_{x\to\infty} \mathbf{P}\left(f(1 - e^{-x(1+\delta(x))}) > y | X > x(1+\delta(x))\right) \\ &= \lim_{x\to\infty} \mathbf{P}\left(f\left(1 - e^{-x}\right) > y | X > x\right).\end{aligned}$$

Next we prove that (i) implies (ii). It is easy to see that for any $\delta(x) \to 0$, for all sufficiently large x,

$$1 - \exp(-x(1+\delta(x)+1/x)) \geq \exp(-\exp(-x(1+\delta(x)))) \geq 1 - \exp(-x(1+\delta(x))),$$

and consequently, by the monotonicity of f,

$$f(1-\exp(-x(1+\delta(x)+1/x))) \geq f(\exp(-\exp(-x(1+\delta(x))))) \geq f(1-\exp(-x(1+\delta(x)))).$$

Now, by (i), taking $\delta(x)$ to be $\delta(x) + 1/x$ and $\delta(x)$ respectively, we have that both the first and the third random variables, conditional on $\{X > x\}$, converge in law to γ. It follows that the middle random variable also converges, implying (ii).

To show (iii) from (ii) we simply take $\delta(x) = -\log(\lambda)/x$. Moreover, (iv) and (iii) are equivalent since $f\left(\exp\left(-\lambda e^{-x}\right)\right)$ is the Laplace transform of $Z_1 e^{-x}$.

Finally we derive (5) from (iii). In fact, for all x sufficiently large,

$$\exp(-\exp(-x)) \geq 1 - \exp(-x) \geq \exp(-e \cdot \exp(-x)).$$

Hence, again by the monotonicity of f,

$$f(\exp(-\exp(-x))) \geq f(1 - \exp(-x)) \geq f(\exp(-e \cdot \exp(-x))).$$

The convergence in (5) then follows from (iii) by taking λ to be 1 and e. ☐

Corollary 10. *Assume conditions* (2) *and* (5)*. Then for any function* $\delta(x)$ *satisfying* $\lim_{x\to\infty}\delta(x) = 0$*,*

$$\mathbf{E}[\gamma] = \lim_{x\to\infty} \mathbf{E}\left[f(1-e^{-x(1+\delta(x))}))|X > x\right] = \lim_{x\to\infty} \mathbf{E}\left[\mathbf{E}_\pi\left[e^{-\lambda Z_1/e^x}\right]\middle| X > x\right]. \quad (14)$$

In particular,

$$\mathbf{E}[\gamma] = \lim_{x\to\infty} \mathbf{P}(Z_1 \le e^{x(1+\delta(x))}|X > x). \quad (15)$$

Proof. This follows by applying the dominated convergence theorem to the convergences in (i) and (iii). □

Lemma 11. *If* (1), (2) *and Assumption* 1 *are valid, then*

$$\lim_{n\to\infty} \mathbf{P}\left(\exp(-na - 2n/\log n) \le \mathbf{P}_\pi(Z_n > 0) \le \exp(-na + n^{2/3})\right) = 1. \quad (16)$$

In particular, for any sequence δ_n *such that* $n(\delta_n - 2/\log n) \to \infty$*,*

$$\lim_{n\to\infty} \mathbf{P}\left(Z_n > 0 \mid Z_0 \ge e^{n(a+\delta_n)}\right) = 1. \quad (17)$$

Proof. The second claim (17) follows directly from the first one, so we shall only prove (16). We have (see, for instance, Geiger and Kersting (2000)) that

$$\mathbf{P}_\pi(Z_n > 0) = \left(e^{-S_n} + \sum_{k=0}^{n-1} g_k(f_{k+1,n}(0))e^{-S_k}\right)^{-1}, \quad (18)$$

where

$$g_k(s) := \frac{1}{1-f_k(s)} - \frac{1}{f_k'(1)(1-s)}$$

meets the estimates

$$0 \le g_k(s) \le 2\eta_{k+1} \quad \text{with} \quad \eta_{k+1} := \eta(\pi_k).$$

Introduce the events

$$G_n := \left\{\max_{1\le k\le n} |S_k + ka| < n^{2/3}\right\} \quad \text{and} \quad H_n := \left\{1 + \sum_{k=1}^{n} \eta_k \le 2ne^{n/\log n}\right\}. \quad (19)$$

By the functional central limit theorem, $\lim_{n\to\infty} \mathbf{P}(G_n) = 1$. Further, by Assumption 1(i),

$$1 - \mathbf{P}(H_n) \le n\mathbf{P}\left(\eta_1 \ge e^{n/\log n}\right) = n \times o\left(\frac{\log n}{n\left(\log\left(n/\log n\right)\right)^{1+\delta}}\right) = o\left(\frac{1}{\log^\delta n}\right).$$

Thus, $\lim_{n\to\infty} \mathbf{P}(H_n) = 1$, and, consequently,

$$\lim_{n\to\infty} \mathbf{P}(G_n \cap H_n) = 1. \tag{20}$$

It then suffices to show that on the event $G_n \cap H_n$,

$$\exp(-na - 2n/\log n) \le \mathbf{P}_\pi(Z_n > 0) \le \exp(-na + n^{2/3})$$

for all sufficiently large n.

The upper bound follows from the evident estimates

$$\mathbf{P}_\pi(Z_n > 0) \le \mathbf{E}_\pi[Z_n] = \exp(S_n) \le \exp(-na + n^{2/3}).$$

As to the lower bound, since $g_k(s) \le 2\eta_{k+1}$, we have

$$\mathbf{P}_\pi(Z_n > 0) \ge \frac{1}{e^{-S_n} + 2\sum_{k=0}^{n-1} \eta_{k+1} e^{-S_k}}.$$

Finally observe that on the event $G_n \cap H_n$ we have

$$\begin{aligned} e^{-S_n} + 2\sum_{k=0}^{n-1} \eta_{k+1} e^{-S_k} &\le 2e^{na+n^{2/3}} \left(1 + \sum_{k=0}^{n-1} \eta_{k+1}\right) \\ &\le 4e^{na+n^{2/3}} \cdot ne^{n/\log n} \le e^{na+2n/\log n} \end{aligned} \tag{21}$$

for all sufficiently large n. □

Lemma 12. *Assume* (1), (2) *and Assumption* 1. *Then for any* $k \in \mathbb{N}$,

$$\lim_{n\to\infty} \mathbf{P}(Z_n > 0 \,|\, X_1 > na, Z_0 = k) = \mathbf{E}\left[1 - \gamma^k\right].$$

Proof. We only prove for the case when $k = 1$. We have

$$\mathbf{P}(Z_n > 0 \,|\, X_1 > na) = \mathbf{E}[\mathbf{P}_\pi(Z_n > 0) \mid X_1 > na] = \mathbf{E}[1 - f_0(f_{1,n}(0)) \,|\, X_1 > na].$$

Write

$$1 - f_{1,n}(0) = \mathbf{P}_\pi(Z_n > 0 | Z_1 = 1) = e^{-an(1+\zeta(n))}.$$

According to the previous lemma, there exists a deterministic function $\delta(n) \to 0$ as $n \to \infty$ such that

$$|\zeta(n)| \le \delta(n)$$

with probability approaching 1 as $n \to \infty$. The conclusion then follows from Corollary 10. □

Lemma 13. *Assume conditions* (1), (2) *and* (3). *Then for any* $\varepsilon > 0$ *there exists* M *such that for all* n *sufficiently large,*

$$\mathbf{P}(Z_n > 0; \tau_n > M) = \mathbf{E}[\mathbf{P}_\pi(Z_n > 0); \tau_n > M] \le \varepsilon \mathbf{P}(X > na).$$

Proof. Using the inequality $\mathbf{P}_\pi(Z_n > 0) \le e^{S_{\tau_n}}$ established in (13), we have by Proposition 4 and Lemma 6 that

$$\begin{aligned}
&\mathbf{E}\left[\mathbf{P}_\pi(Z_n > 0); \tau_n > M\right] \le \sum_{k=M}^{n} \mathbf{E}\left[e^{S_{\tau_n}}; \tau_n = k\right] \\
= & \sum_{M\le k\le n/2} \mathbf{E}\left[e^{S_k}; \tau_k = k\right] \mathbf{P}\left(L_{n-k} \ge 0\right) + \sum_{n/2<k\le n} \mathbf{E}\left[e^{S_k}; \tau_k = k\right] \mathbf{P}\left(L_{n-k} \ge 0\right) \\
\le & \ \mathbf{P}\left(L_{[(n+1)/2]} \ge 0\right) \sum_{k=M}^{\infty} \mathbf{E}\left[e^{S_k}; \tau_k = k\right] + C\frac{\mathbf{P}\left(X > an\right)}{n} \sum_{k\le n/2} \mathbf{P}\left(L_k \ge 0\right) \\
\le & \ \varepsilon \mathbf{P}\left(X > an\right).
\end{aligned}$$

□

Now we are ready to prove Theorem 2.

Proof of Theorem 2. For fixed M and N we write

$$\begin{aligned}
\mathbf{P}\left(Z_n > 0\right) &= \mathbf{P}\left(Z_n > 0; U_n \le NM\right) \\
&\quad + \mathbf{P}\left(Z_n > 0; U_n > NM; \tau_n > M\right) + \mathbf{P}\left(Z_n > 0; U_n > NM; \tau_n \le M\right).
\end{aligned}$$

By Lemma 13, for any $\varepsilon > 0$ there exists M such that for all sufficiently large n,

$$\mathbf{P}\left(Z_n > 0; U_n > NM; \tau_n > M\right) \le \mathbf{P}\left(Z_n > 0; \tau_n > M\right) \le \varepsilon \mathbf{P}\left(X > na\right).$$

Moreover, by Propositions 8 and 4 there exists N such that for all sufficiently large n,

$$\begin{aligned}
\mathbf{P}\left(Z_n > 0; U_n > NM; \tau_n \le M\right) &\le \mathbf{P}\left(U_n > NM; \tau_n \le M\right) \\
&= \sum_{k=0}^{M} \mathbf{P}\left(U_n > NM; \tau_n = k\right) \\
&\le \sum_{k=0}^{M} \mathbf{P}\left(\tau_k = k\right) \mathbf{P}\left(U_n > (N-1)M; \tau > n - M\right) \\
&\le (M+1)\mathbf{P}\left(U_n > (N-1)M; \tau > n - M\right) \\
&\le \varepsilon \mathbf{P}\left(\tau > n - M\right) \le 2e^D \varepsilon \mathbf{P}\left(X > na\right).
\end{aligned}$$

Hence the main contribution to $\mathbf{P}\left(Z_n > 0\right)$ comes from $\mathbf{P}\left(Z_n > 0; U_n \le NM\right)$, i.e., when there is a big jump of the associated random walk at the beginning.

To proceed, we introduce the events

$$A_k = A_k(n) := \left\{X_i \le na, 1 \le i \le k\right\}, \quad k = 1, 2, \dots, n. \tag{22}$$

For each fixed j, we have by the Markov property

$$\begin{aligned}
& \mathbf{P}(Z_n > 0; U_n = j) \\
= \;& \mathbf{P}(A_{j-1}; X_j > na; Z_n > 0) \\
= \;& \sum_{k=1}^{\infty} \mathbf{P}(A_{j-1}; Z_{j-1} = k)\mathbf{P}(X_1 > na)\mathbf{P}\left(Z_{n-j+1} > 0 | X_1 > na; Z_0 = k\right). \quad (23)
\end{aligned}$$

Clearly, $\mathbf{P}(A_{j-1}, Z_{j-1} = k) = \mathbf{P}(A_{j-1}(n), Z_{j-1} = k)$ increases to $\mathbf{P}(Z_{j-1} = k)$ as $n \to \infty$. Dividing both sides of (23) by $\mathbf{P}\left(X_1 > na\right)$ and applying the dominated convergence theorem and Lemma 12 yield

$$\begin{aligned}
\frac{\mathbf{P}\left(Z_n > 0; U_n = j\right)}{\mathbf{P}\left(X_1 > na\right)} &= \sum_{k=1}^{\infty} \mathbf{P}\left(Z_{j-1} = k; A_{j-1}\right)\mathbf{P}\left(Z_{n-j+1} > 0 | X_1 > na, Z_0 = k\right) \\
&\sim \sum_{k=1}^{\infty} \mathbf{P}\left(Z_{j-1} = k\right)\mathbf{E}\left[1 - \gamma^k\right] \\
&= \mathbf{E}\left[1 - f_{0,j-1}\left(\gamma\right)\right]. \qquad (24)
\end{aligned}$$

The last equality holds because of the independence assumption that we put on γ and $\{f_{0,j}\}$. Hence

$$\mathbf{P}\left(Z_n > 0\right) \sim \sum_{j=1}^{\infty} \mathbf{E}\left[1 - f_{0,j-1}\left(\gamma\right)\right] \cdot \mathbf{P}\left(X > na\right) \qquad (25)$$

as desired. □

4. Functional Limit Theorems Conditional on Non-extinction

The proof of Theorem 2 shows that the process survives owing to one big jump of the associated random walk which happens at the very beginning of the evolution of the process. This motivates the study of the conditional distribution of U_n given $\{Z_n > 0\}$, which is the content of the next lemma.

Lemma 14. *For any $j \geq 1$,*

$$\lim_{n\to\infty} \mathbf{P}(U_n = j | Z_n > 0) = \mathbf{E}\left[1 - f_{0,j-1}(\gamma)\right]/K, \qquad (26)$$

where K is as in Theorem 2. In particular, for any $\varepsilon > 0$, there exists $M > 0$ such that

$$\limsup_{n\to\infty} \mathbf{P}(U_n > M | Z_n > 0) \leq \varepsilon. \qquad (27)$$

Proof. The first claim follows from the representation

$$\mathbf{P}(U_n = j | Z_n > 0) = \frac{\mathbf{P}(Z_n > 0; U_n = j)}{\mathbf{P}(X_1 > na)} \times \frac{\mathbf{P}(X_1 > na)}{\mathbf{P}(Z_n > 0)}$$

and relationships (24) and (25).

Estimate (27) follows from (26) and Corollary 7. □

Thus, we have demonstrated that given survival to time n, there must be a big jump at the early time period. Next lemma complements this by showing that for survival of the process such a big jump will be accompanied by a population explosion.

Let $Z_j(i)$ be the offspring size of the i-th particle existing in generation $j-1$, and, as we shall deal with $\max_{1\le i\le Z_{U_n-1}} Z_{U_n}(i)$ repeatedly, define

$$N_{U_n} := \max_{1\le i\le Z_{U_n-1}} Z_{U_n}(i). \tag{28}$$

Lemma 15. *For any sequence h_n such that $h_n \le n$ and $h_n \to \infty$, and $\delta_n \to 0$ satisfying $n(\delta_n - 2/\log n) \to \infty$,*

$$\lim_{n\to\infty} \mathbf{P}\left(U_n < h_n, N_{U_n} \ge e^{n(a+\delta_n)} \,|\, Z_n > 0\right) = 1.$$

Proof. We first estimate the probability

$$\begin{aligned} & \mathbf{P}\left(U_n < h_n, N_{U_n} \ge e^{n(a+\delta_n)}, Z_n > 0\right) \\ = & \sum_{j<h_n}\sum_{k=1}^{\infty} \mathbf{P}(U_n = j, Z_{j-1} = k)\cdot \mathbf{P}\left(N_{U_n} \ge e^{n(a+\delta_n)} \,|\, U_n = j, Z_{j-1} = k\right) \\ & \cdot \mathbf{P}(Z_n > 0 \,|\, N_{U_n} \ge e^{n(a+\delta_n)}, U_n = j, Z_{j-1} = k). \end{aligned} \tag{29}$$

Recall the events $A_j(n)$ that we defined in (22). As $n\to\infty$,

$$\mathbf{P}(U_n = j, Z_{j-1} = k) = \mathbf{P}(Z_{j-1} = k, A_{j-1}(n))\cdot\mathbf{P}(X > an) \sim \mathbf{P}(Z_{j-1} = k)\cdot\mathbf{P}(X > an).$$

Moreover, by (a simple generalization of) (15), for any fixed j and k,

$$\lim_{n\to\infty} \mathbf{P}\left(N_{U_n} \ge e^{n(a+\delta_n)} \,|\, U_n = j, Z_{j-1} = k\right) = \mathbf{E}[1-\gamma^k].$$

Finally, by (17) in Lemma 11,

$$\mathbf{P}(Z_n > 0 \,|\, N_{U_n} \ge e^{n(a+\delta_n)}, U_n = j, Z_{j-1} = k) \to 1.$$

Dividing both sides of (29) by $\mathbf{P}(Z_n > 0)$ and applying Theorem 2 and Fatou's lemma we get the conclusion. □

The arguments above lead to the following lemma, which says that conditioning on $\{Z_n > 0\}$ is asymptotically equivalent to conditioning on $\{U_n < h_n, N_{U_n} \ge e^{n(a+\delta_n)}\}$.

Lemma 16. *For any sequence h_n such that $h_n \le n$ and $h_n \to \infty$, and $\delta_n \to 0$ satisfying $n(\delta_n - 2/\log n) \to \infty$,*

$$||\mathbf{P}(\cdot\,|\, Z_n > 0) - \mathbf{P}(\cdot\,|\, U_n < h_n, N_{U_n} \ge e^{n(a+\delta_n)})||_{TV} \to 0, \tag{30}$$

where $||\cdot||_{TV}$ denotes the total variation distance.

Proof. This follows from Lemma 17 in Lalley and Zheng (2011), relation (17), and the previous lemma. □

Lemma 16 shows that to prove the functional limit theorems for the population size up to moment n conditioned on survival of the process to this moment, we need only to establish limit theorems under the condition $\{U_n < h_n, N_{U_n} \geq e^{n(a+\delta_n)}\}$.

Lemma 17. *For all sufficiently large n, on the event $G_n \cap H_n$ (as defined in (19)) we have*

$$\frac{\mathbf{E}_\pi\left[Z_n^2\right]}{\exp(2S_n)} \leq \exp(na + 2n/\log n).$$

Proof. Recall the generating functions $f_{k,n}(\cdot)$'s that we defined in (6). Clearly,

$$f'_{0,n}(s) = \prod_{k=0}^{n-1} f'_k(f_{k+1,n}(s)),$$

and

$$f''_{0,n}(s) = \prod_{i=0}^{n-1} f'_i(f_{i+1,n}(s)) \cdot \sum_{k=0}^{n-1} \frac{f''_k(f_{k+1,n}(s))}{f'_k(f_{k+1,n}(s))} \prod_{j=k+1}^{n-1} f'_j(f_{j+1,n}(s)).$$

Hence, letting $s = 1$ we get

$$\mathbf{E}_\pi\left[Z_n(Z_n - 1)\right] = f''_{0,n}(1) = 2e^{2S_n} \cdot \sum_{k=0}^{n-1} \eta_{k+1} e^{-S_{k+1}}.$$

Thus,

$$\mathbf{E}_\pi\left[Z_n^2\right] = f''_{0,n}(1) + \mathbf{E}_\pi\left[Z_n\right] = 2e^{2S_n} \cdot \sum_{k=0}^{n-1} \eta_{k+1} e^{-S_{k+1}} + e^{S_n}$$

implying

$$\frac{\mathbf{E}_\pi\left[Z_n^2\right]}{\exp(2S_n)} = 2 \cdot \sum_{k=0}^{n-1} \eta_{k+1} e^{-S_{k+1}} + e^{-S_n}.$$

The needed conclusion then follows from an argument similar to (21). □

For the following two lemmas we fix a sequence h_n such that

$$h_n \to \infty, \quad \text{and} \quad h_n/n \to 0. \tag{31}$$

A simple example of such a choice is $h_n = \log n$. We also take δ_n to be a sequence satisfying

$$\delta_n \to 0 \quad \text{and} \quad n(\delta_n - 2/\log n) \to \infty. \tag{32}$$

Lemma 18. *Suppose* (31) *and* (32) *hold. Then*

$$\mathcal{L}\left(\frac{Z_{[nt]\vee U_n}}{Z_{U_n}\exp(S_{[nt]\vee U_n}-S_{U_n})}, 0\le t\le 1 \,\middle|\, U_n<h_n, N_{U_n}\ge e^{n(a+\delta_n)}\right) \Longrightarrow (1, 0\le t\le 1)\,.$$

Proof. Let $\pi'=\{\pi_{U_n},\pi_{U_n+1},\ldots\}$ be the random environment after time U_n, and $S'_m := S_{m+U_n}-S_{U_n}$ be the random walk after time U_n. Further, define G'_n and H'_n for the random environment π' in the same way as in (19). Then, by (20), as $n\to\infty$, $G'_n\cap H'_n$ occurs with probability approaching one, so we need only to prove the convergence on the event $G'_n\cap H'_n$.

We first prove the marginal convergence, by a mean-variance calculation. Denote $k=[nt]$. By our assumption $U_n\le h_n$ for an $h_n/n\to 0$, implying $k=[nt]>U_n$ for all $t>0$ and for all n big enough. Hence

$$\mathbf{E}_\pi(Z_k|Z_{U_n}) = Z_{U_n}\cdot\exp(S'_{k-U_n});$$

moreover, by Lemma 17, for all n big enough, on the event $G'_n\cap H'_n$,

$$\mathrm{Var}_\pi(Z_k|Z_{U_n}) = Z_{U_n}\cdot \mathrm{Var}_{\pi'}(Z_{k-U_n}|Z_0=1)\le Z_{U_n}\cdot\exp(2S'_{k-U_n})\cdot\exp(na+2n/\log n).$$

Consequently, when $N_{U_n}\ge\exp(n(a+\delta_n))$ and, therefore, $Z_{U_n}\ge\exp(n(a+\delta_n))$, we have

$$\mathrm{Var}_\pi\left(\frac{Z_k}{Z_{U_n}\cdot\exp(S'_{k-U_n})}\,\middle|\, Z_{U_n}\right)\le\frac{\exp(na+2n/\log n)}{Z_{U_n}}\le\frac{\exp(na+2n/\log n)}{\exp(n(a+\delta_n))}\to 0.$$

Next, by Slutsky's theorem (see, e.g., Ferguson (1996)) we have convergence of finite dimensional distributions. Furthermore, since $Z_{[nt]\vee U_n}/(Z_{U_n}\exp(S'_{[nt]\vee U_n-U_n}))$ are martingales (with respect to the post-U_n sigma field $\mathcal{F}^\pi_{([nt]\vee U_n)}$, where $\mathcal{F}^\pi_i=\sigma\langle\pi;Z_j,j\le i\rangle$), to prove the convergence in the space $D[0,1]$ we need only to show, by Proposition 1.2 in Aldous (1989), the uniform integrability of $\{Z_{[nt]\vee U_n}/(Z_{U_n}\exp(S'_{[nt]\vee U_n-U_n}))\}$ for any fixed t. This follows from the above calculation, demonstrating that the elements of the martingale sequence in question are bounded in L^2. ☐

The lemma just proved serves as an LLN type result; the following lemma gives the CLT type statement.

Lemma 19. *Suppose* (31) *and* (32) *hold. Then*

$$\mathcal{L}\left(\frac{1}{\sigma\sqrt{n}}\left(\log\left(Z_{[nt]\vee U_n}/Z_{U_n}\right)+nta\right), 0\le t\le 1\,\middle|\, U_n\le h_n, N_{U_n}\ge e^{n(a+\delta_n)}\right) \Longrightarrow (B_t, 0\le t\le 1),$$

and for any $\varepsilon > 0$,

$$\mathcal{L}\left(\frac{1}{\sigma\sqrt{n}}\left(\log\left(Z_{[nt]}/Z_{[n\varepsilon]}\right) + n(t-\varepsilon)a\right), \varepsilon \le t \le 1 \,\middle|\, U_n \le h_n, N_{U_n} \ge e^{n(a+\delta_n)}\right) \Longrightarrow (B_t - B_\varepsilon, \varepsilon \le t \le 1),$$

where σ *is the standard deviation of* X, *and* B_t *is a standard Brownian motion.*

Proof. By Lemma 18 and the continuous mapping theorem,

$$\mathcal{L}\left(\log\left(Z_{[nt]\vee U_n}/Z_{U_n}\right) - (S_{[nt]\vee U_n} - S_{U_n}), 0 \le t \le 1 \,\middle|\, U_n \le h_n, N_{U_n} \ge e^{n(a+\delta_n)}\right) \Longrightarrow (0, 0 \le t \le 1)\,.$$

The conclusions then follow from the standard functional central limit theorem for the post-U_n random walk $\{S_{[nt]\vee U_n} - S_{U_n}\}$. □

Now we are ready to prove Theorem 3.

Proof of Theorem 3. The first claim follows from Lemma 14. The second statement follows from Lemmas 16, 18 and 19. □

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